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ON THE STABILITY OF STEADY ROTATION OF A CYLINDER PARTLY FILLED WITH A VISCOUS INCOMPRESSIBLE FLUID*

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The problem of stability in the small of steady rotation at constant angular velocity of a cylinder partly filled with viscous incompressible fluid and its axis held in viscoelastic supports is considered within the limits of a plane model. Taking into account the problem symmetry we reduce the conditions that define parameters for which a change of the system degree of instability occurs to conditions of existence of solutions of the equations in variations that define the circular precession of a cylinder containing fluid. The derived exact solution of the hydrodynamic problem defines the forces exerted by the viscous incompressible fluid that partly fills the rotating cylinder on the latter under conditions of circular precession. Expressions for components of that force are used for dividing the parameter plane of the cylinder axis supports into regions of different degrees of instability.

In analyzing models of turbomachinery we encounter the problem of stability of steady rotation of a circular cylinder partly filled with a viscous incompressible fluid and held axisymmetrically in viscoelastic supports under conditions of constancy of the cylinder angular velocity projections on the axis of steady rotation. Only a small number of publications /1 – 3/ dealt with the dynamics of bodies with cavities containing a fluid.

1. Statement of the problem. Let a circular cylinder of radius a steadily rotate about its axis (which coincides with the Z axis of the fixed Cartesian coordinate system $Oxyz$) mounted in viscoelastic axisymmetric supports. The viscous incompressible fluid which partly fills the cylinder forms a layer of constant thickness h on the inner surface of the cylinder, rotating with the latter as a solid body. We shall consider the problem of stability in linear approximation using a plane model, i.e. we assume that points of the cylinder can only move in a direction parallel to the Oxy plane, and that the fluid velocity field has only x, y components which, as well as the fluid pressure are independent of Z. The plane model is admissible, if the cylinder axial displacement and its axis angular displacements are negligibly small (e.g., a cylinder with its axis in bearings) and the cylinder is fairly long (end effects are negligible).

The system of linearized equations for a plane model and boundary conditions used here consists of the following.

1) Equations of translational motion of the cylinder parallel to the Oxy plane, linearized close to the cylinder state in steady rotation

$$Mx_0^{\cdot\cdot} + Hx_0^{\cdot} + Kx_0 = F_x, \quad My_0^{\cdot\cdot} + Hy_0^{\cdot} + Ky_0 = F_y \tag{1.1}$$

where x_0, y_0 are the coordinates of the intersection point of the cylinder axis with the plane Oxy, F_x, F_y are components of the force exerted by the fluid per unit of cylinder length, M is the mass of a unit length of the cylinder, and H and K are the coefficients, respectively, of damping and rigidity of the cylinder axis supports, per unit of its length.

2) The condition of constancy of the cylinder angular velocity of rotation about the Z axis: $\Omega = \text{const}$.

3) Equations of motion of the viscous incompressible fluid in the Oxy plane and linearized close to the steady state of quasi-solid rotation of the fluid about the Oz axis.

4) Conditions of fluid adherence to the cylinder surface, of continuity of stresses, and the kinematic condition at the free surface transferred in linear approximation with respect to deviations from steady rotation, onto surfaces $x^2 + y^2 = a^2$ and $x^2 + y^2 = (a - h)^2$, respectively.

5) Formulas that define F_x, F_y in terms of deviations of pressure and of the fluid velocity field components from their respective values steady quasi-solid rotation.

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2. Properties of symmetry and the circular precession. The indicated equations and boundary conditions are linear and homogeneous with respect to deviations from the steady state rotation of the cylinder and of fluid that partly fills its cavity, and obviously have the following two symmetry properties: a) invariance to shifts of the time reference point, i.e. to transformation $t' = t - t_0$, and b) invariance to turning of the coordinate system about Oz by the angle $\pi/2$, i.e. to transformation $x' = y, y' = -x, z' = z$.

By virtue of the symmetry property a) the system of Eqs.1)—5) admits the particular solutions $\sim e^{\lambda t}$, where λ is the eigenvalue. We shall consider the rotation of a cylinder containing fluid as steady in the small, when all λ have negative real parts, and unsteady, if even a single λ has a positive real part. If the eigenvalues λ continuously depend on parameters of the problem, a change of the system degree of instability takes place at the appearance of an imaginary $\lambda = i\omega$. Then besides solution

$$(x^*\mathbf{e}_x + y^*\mathbf{e}_y,\ v_x^*(x, y)\,\mathbf{e}_x + v_y^*(x, y)\,\mathbf{e}_y,\ p^*(x, y))\,e^{i\omega t}$$

of the system of Eqs.1)—5) we have by virtue of the symmetry property b) also the solution

$$(-y^*\mathbf{e}_x + x^*\mathbf{e}_y,\ -v_y^*(y, -x)\,\mathbf{e}_x + v_x^*(y, -x)\,\mathbf{e}_y,\ p^*(y, -x))\,e^{i\omega t}$$

where x^*, y^* are the complex amplitudes of the radius vector component of the point of intersection of the cylinder axis with the Oxy plane, and v_x^*, v_y^*, p^* are the complex amplitudes of deviations component, of the fluid velocity and pressure fields from their respective values in the steady quasi-solid rotation.

Multiplying the first of these solutions by i and adding it to the second, we obtain by virtue of the linearity of Eqs.1)—5) a particular solution that describes the so-called circular precession of a cylinder containing fluid, i.e. a motion in which the intersection point of the cylinder axis with the Oxy plane describes a circle and the deviation of hydrodynamic quatities from their steady state values changes in time $\sim e^{i\omega t}$. Conversely, if for some values of parameters Eqs.1)—5) admit solutions of the circular precession type, there exists an imaginary eigenvalue λ. Thus parameter values for which a change of the instability degree occurs (when at least one eigenvalue is imaginary) can be obtained from the condition of existence circular precession of the cylinder containing fluid.

The above consideration determines the process used here for solving this problem. First, we consider the hydrodynamic problem of motion of a viscous incompressible fluid partly filling a rotating cylinder performing circular precession. Then we calculate the force exerted by the fluid on a rotating cylinder in the case of circular precession. Finally, using the obtained expressions for the hydrodynamic force and from Eq.(1.1) of the cylinder translational motion, we obtain the conditions under which circular precession is possible. These conditions determine, in conformity with the above analysis, the boundaries of regions with different degrees of instability in the parameter field of the problem.

3. The hydrodynamic problem. Let an infinitely long circular cylinder of inner radius a rotates about its axis at angular velocity ω_0 and process at frequency ω, so that its axis describes a cylindrical surface of radius ε (Fig.1). The cylinder absolute angular velocity Ω is the sum of the cylinder proper angular velocity ω_0 and of the precession velocity ω $(\Omega = \omega_0 + \omega)$.

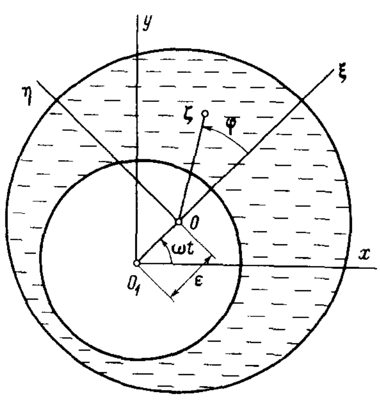

Fig.1

We choose the noninertial reference system $O\xi\eta$ (Fig. 1) rigidly attached to the so-called line of centers which passes through the precession center O_1 and the cylinder cross section center O. This system translates along the circle of radius ε at velocity $\omega\varepsilon$ and rotates about the cylinder axis at angular velocity ω. We introduce in the reference space $O\xi\eta$ the system of polar coordinates r, φ with the center at point O. In this coordinate system components of the field of inertia forces acting on a particle of unit mass are in the reference space $O\xi\eta$ of the form

$$f_r = \omega^2 r + 2\omega v + \omega^2\varepsilon\cos\varphi,\quad f_\varphi = -2\omega u - \omega^2\varepsilon\sin\varphi \tag{3.1}$$

where u, v are the radial and azimuthal components of particle velocity relative to the reference system $O\xi\eta$.

Consider the problem of plane motion of a viscous incompressible fluid partly filling a cylinder rotating with a circular precession of small radius ε, for which deviations of hydrodynamic quantities from their steady state values are small and vary in the fixed reference system in time as $\sim e^{i\omega t}$. Using the law of energy variation of a viscous fluid, it is possible to show that in the case of circular precession of small radius the fluid motion is stable in the reference system $O\xi\eta$, i.e. is time independent.

The equations of the fluid stable motion relative to the system $O\xi\eta$ and the boundary conditions are linearized near the steady quasi-solid rotation of fluid about the cylinder axis

$$u = 0, \; v = \omega_0 r \tag{3.2}$$

are of the form

$$\omega_0 \frac{\partial u'}{\partial \varphi} = \Omega^2 r + \omega^2 \varepsilon \cos\varphi + 2\Omega v' - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u' - \frac{u'}{r^2} - \frac{2}{r^2}\frac{\partial v'}{\partial \varphi}\right) \tag{3.3}$$

$$\omega_0 \frac{\partial v'}{\partial \varphi} = -\omega^2 \varepsilon \sin\varphi - 2\Omega u' - \frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\Delta v' - \frac{v'}{r^2} + \frac{2}{r^2}\frac{\partial u'}{\partial \varphi}\right)$$

$$\frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{1}{r}\frac{\partial v'}{\partial \varphi} = 0$$

$$u' = 0, \; v' = 0, \; r = a \tag{3.4}$$

$$-p - \rho\Omega^2 r\eta + 2\mu\frac{\partial u'}{\partial r} = -p_0, \quad \frac{\partial v'}{\partial r} + \frac{1}{r}\frac{\partial u'}{\partial \varphi} - \frac{v'}{r} = 0, \quad \omega_0\, \partial\eta/\partial\varphi = u', \; r = a - h \tag{3.5}$$

where u', v' are small deviations of the velocity field components from (3.2) p is the pressure, ρ is the density ν, μ are, respectively, the kinematic and dynamic viscosities of the fluid, $r = a - h + \eta(\varphi)$ is the equation of the fluid free surface, and p_0 is the pressure on the free surface.

We introduce Lamb's potentials θ, ψ and the function χ

$$u' = \frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\varphi}, \quad v' = \frac{1}{r}\frac{\partial\theta}{\partial\varphi} - \frac{\partial\psi}{\partial r}$$

$$\chi = {}^1/_2\Omega^2 r^2 + \omega^2 \varepsilon r \cos\varphi - 2\Omega\psi + \text{const}$$

System (3.3) can then be represented in the form

$$\frac{\partial}{\partial r}F + \frac{1}{r}\frac{\partial}{\partial\varphi}G = 0, \quad \frac{1}{r}\frac{\partial}{\partial\varphi}F - \frac{\partial}{\partial r}G = 0, \quad \Delta\theta = 0 \tag{3.6}$$

$$F = \chi - \frac{p}{\rho} - \omega_0\frac{\partial\theta}{\partial\varphi}, \quad G = \nu\Delta\psi + 2\Omega\theta - \omega_0\frac{\partial\psi}{\partial\varphi}$$

As shown in the Appendix, the ambiguity in the selection of Lamb's potentials (gauging of potentials) can be dealt with in such a way that (3.6) reduces to the system

$$F = 0, \; G = 0, \; \Delta\theta = 0 \tag{3.7}$$

After the introduction of Lamb's potentials, boundary conditions (3.4) and (3.5) assume the form

$$\frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\varphi} = 0, \quad \frac{1}{r}\frac{\partial\theta}{\partial\varphi} - \frac{\partial\psi}{\partial r} = 0, \quad r = a \tag{3.8}$$

$$2\nu\left(\frac{\partial^2\theta}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial\varphi} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\,\partial\varphi}\right) - \frac{p}{\rho} - \Omega^2 r\eta = -\frac{p_0}{\rho} \tag{3.9}$$

$$r^2\frac{\partial^2\psi}{\partial r^2} - 2r\frac{\partial^2\theta}{\partial r\,\partial\varphi} - \frac{\partial^2\psi}{\partial\varphi^2} + 2\frac{\partial\theta}{\partial\varphi} - r\frac{\partial\psi}{\partial r} = 0$$

$$\omega_0\frac{\partial\eta}{\partial\varphi} = \frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\varphi}, \quad r = a - h$$

Equations (3.7) with boundary condtions (3.8) and (3.9) contain only the following dimensional parameters: $\omega_0, \Omega, \nu, a, a-h, \varepsilon$ (parameter p_0 is immaterial, since the fluid is incompressible). Owing to the linearity of the formulated above boundary value problem, parameter ε appears in the solution in the first power. It only defines the scale of velocity of the fluid motion induced by the cylinder precession. The remaining five parameters constitute only three independent dimensionless combinations

$$\frac{\omega}{\Omega}, \quad \frac{a-h}{a}, \quad \frac{\nu}{\Omega a^2} \tag{3.10}$$

which in this problem are the similarity criteria.

4. Determination of the hydrodynamic force. We pass to the solution of the boundary value problem (3.7)—(3.9), and shall seek a solution of system (3.7) of the form

$$\theta = 2\,\mathrm{Re}\,[\Theta(r)e^{i\varphi}], \quad \psi = 2\,\mathrm{Re}\,[\Psi(r)e^{i\varphi}], \quad i^2 = -1$$

From the third equation of system (3.7) we have

$$\theta = 2\,\mathrm{Re}\left[\left(c_1 r + \frac{c_2}{r}\right)e^{i\varphi}\right] \tag{4.1}$$

after which the second equation reduces to

$$\frac{d^2\Psi}{dr^2} + \frac{1}{r}\frac{d\Psi}{dr} - \left(\frac{i\omega_0}{\nu} + \frac{1}{r^2}\right)\Psi = -\frac{2\Omega}{\nu}\left(c_1 r + \frac{c_2}{r}\right) \tag{4.2}$$

Integrating (4.2) we obtain

$$\psi = 2\,\mathrm{Re}\left\{\left[-\frac{2\Omega}{\omega_0} i\left(c_1 r + \frac{c_2}{r}\right) + c_3 L_1(kr) + c_4 M_1(kr)\right]e^{i\varphi}\right\} \tag{4.3}$$

$$L_1 = e^{-\varkappa a} H_1^{(2)}(kr), \quad M_1 = e^{\varkappa b} H_1^{(1)}(kr)$$

$$k = \varkappa\left(-\frac{\omega_0}{|\omega_0|} + i\right), \quad \varkappa = \sqrt{\frac{|\omega_0|}{2\nu}}, \quad b = a - h$$

where $(H_n^{(1),(2)}(kr)$ is the Hankel function.

From the first of equations (3.7) for pressure we obtain

$$\frac{p}{\rho} = 2\,\mathrm{Re}\left\{\left[-i\omega_0\left(c_1 r + \frac{c_2}{r}\right) + \frac{\omega^2\varepsilon r}{2}\right]e^{i\varphi}\right\} - 2\Omega\psi + \frac{\Omega^2 r^2}{2} + C \tag{4.4}$$

Definition of the fluid free surface deflection $\eta(\varphi)$ is sought in the form

$$\eta(\varphi) = 2\mathrm{Re}\,(\eta^* e^{i\varphi})] \tag{4.5}$$

The substitution (4.1) and (4.3) — (4.5) into the boundary conditions (3.8) and (3.9) yields a system of linear algebraic equations for constants c_1, c_2, c_3, c_4, as well as expressions for η^* and the additive constant in (4.4)

$$\frac{3-\tau}{1-\tau}c_1 + \frac{1+\tau}{1-\tau}\frac{c_2}{a^2} + \frac{i}{a}Z_1(ka) = 0 \tag{4.6}$$

$$\frac{3-\tau}{1-\tau}ic_1 - \frac{1+\tau}{1-\tau}\frac{i}{a^2}c_2 - kZ_0(ka) + \frac{1}{a}Z_1(ka) = 0$$

$$\frac{1+\tau}{1-\tau}\frac{4i}{b^3}c_2 + \frac{2k}{b}Z_0(kb) + \left(k^2 - \frac{4}{b^2}\right)Z_1(kb) = 0$$

$$-\frac{\tau^2(3-\tau)}{(1-\tau)^2}ibc_1 + \frac{i}{b}(1+\tau)\left[\frac{2-4\tau+\tau^2}{(1-\tau)^2} - \frac{4}{k^2b^2}\right]c_2 -$$

$$2\frac{1-\tau}{kb}Z_0(kb) + \left(\frac{2\tau-1}{1-\tau} + 4\frac{1-\tau}{k^2b^2}\right)Z_1(kb) = -\frac{1}{2}\tau^2\Omega\varepsilon b$$

$$i\omega_0\eta^* = \frac{3-\tau}{1-\tau}c_1 + \frac{1+\tau}{1-\tau}\frac{c_2}{b^2} + \frac{i}{b}Z_1(kb), \quad C = \frac{p_0}{\rho} - \frac{\Omega^2 b^2}{2} \tag{4.7}$$

$$\tau = \frac{\omega}{\Omega}, \quad Z_n(kr) = c_3 L_n(kr) + c_4 M_n(kr)$$

$$L_n(kr) = e^{-\varkappa a}H_n^{(2)}(kr), \quad M_n(kr) = e^{\varkappa b}H_n^{(1)}(kr)$$

In the derivation of (4.6) well known formulas for derivatives of cylindrical functions /4/ were used. In a number of interesting cases the value of $kr\ (b \leqslant r \leqslant a)$ is very large, which enables us to use in (4.6) the asymptotic expansions for these functions. Renormalized Hankel functions $L_n(kr)$, $M_n(kr)$ are very convenient for this purpose.

After constants c_1, c_2, c_3, c_4 have been determined, the boundary value problem (3.7) — (3.9) is virtually solved. Let us calculate the force with which the fluid acts on the cylinder. Integrating the stresses acting on the inner surface of the cylinder, we obtain for components of the force acting per unit of its length

$$F_\xi = 2\pi a\rho\,\mathrm{Re}\left[\frac{1}{2}\omega^2\varepsilon a + 2i(\Omega + \omega)\frac{c_2}{a}\right] \tag{4.8}$$

$$F_\eta = -4\pi\rho\,(\Omega + \omega)\mathrm{Re}\,c_2$$

When the precession frequency $\omega \to \Omega$, it is comparatively simple to obtain the expression for c_2 in (4.6), using the asymptotic expansions for cylindrical functions in the case of small values of the argument /4/, and write (4.8) in the form

$$F_\xi = \pi\rho\Omega^2 a^2\varepsilon + O(\omega_0) \tag{4.9}$$

$$F_\eta = \frac{8\pi\varepsilon\mu\omega_0(\delta^4+1)}{\delta^4 - 1 - 2(\delta^4+1)\ln\delta} + O(\omega_0^2\ln|ka|), \quad \delta = \frac{b}{a}$$

Expressions (4.9) show that in the neighborhood of resonance $\omega \approx \Omega$ the projection of force on the line of centers $F_\xi > 0$, i.e. the hydrodynamic force tends to pull the cylinder axis away from the precession axis (in Fig.1 the intersection point of the precession axis and the plane of the diagram is the precession center O_1). Morever, the component F_η of the force can be nonzero only in the case of a viscous fluid ($\mu \neq 0$). Since the denominator in the expression for F_η when $0 < \delta < 1$ is positive, hence when $\omega < \Omega$ we have $F_\eta > 0$, i.e. the hydrodynamic force tends to increase the angular velocity of cylinder precession and, when $\omega > \Omega$ we have $F_\eta < 0$, which shows the inverse effect of the force. These conclusions are in agreement with the concept of the so-called rotating friction /5/ extensively used in applied investigations. Note, also, that the moment of the hydrodynamic force (4.9) relative to the cylinder axis is zero.

An example shown in Fig.2 of the dependence of dimensionless hydrodynamic force components $F_{*\xi} = F_\xi/F^\circ$ (solid lines) and $F_{*\eta} = F_\eta/F^\circ$ (dash lines) on ω/Ω, calculated for the case of $\delta = 0.5$; $\nu/(\Omega a^2) = 10^{-5}$ using (4.6) and (4.8). The scale of force $F^\circ = m\omega^2\varepsilon$, where $m = \pi\rho(a^2 - b^2)$ is the mass of fluid per unit of cylinder length. The dependence of force on the frequency ratio ω/Ω has a clearly expressed resonance character which is due to resonance excitation of waves propagating over the free surface of the rotating fluid contained in the cylinder.

Comparison of results of calculations of the hydrodynamic force with the force determined with the use of the conservative model ($\mu = 0$) shows a good quantitative agreement between the ξ components outside the neighborhood of resonance values of ω/Ω. However, in the resonance neighborhood the ξ component of the hydrodynamic force, unlike that determined using the conservative model, is finite and comparable in magnitude to the η component. It is also important to point out that in the resonance neighborhood, even in the case of a very small parameter $\nu/(\Omega a^2)$, the wave motion induced by the cylinder precession considerably differs throughout the fluid from the motion determined by the conservative model.

5. Determination of the stability region of steady rotation in the plane of parameters of the cylinder axis supports. We substitute the determined in Sect.4 hydrodynamic force that acts on a unit of cylinder length into the equations of motion of the cylinder, setting in the right-hand sides of (1.1)

$$F_x = F_\xi \cos\omega t - F_\eta \sin\omega t, \quad F_y = F_\xi \sin\omega t + F_\eta \cos\omega t$$

Then, setting in (1.1) $x_0 = \varepsilon\cos\omega t$, $y_0 = \varepsilon\sin\omega t$ to make it conform to circular precession with frequency ω and radius ε, we obtain the relations that link ω and the problem parameters in the case of circular precession

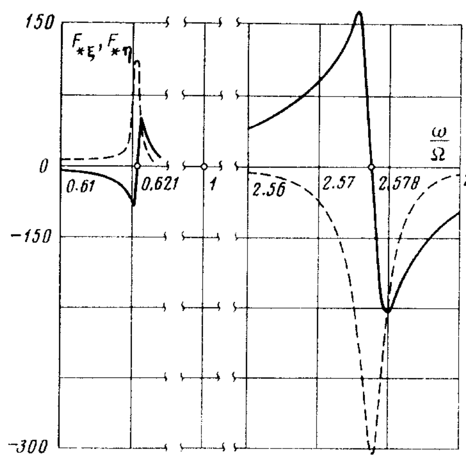

Fig.2

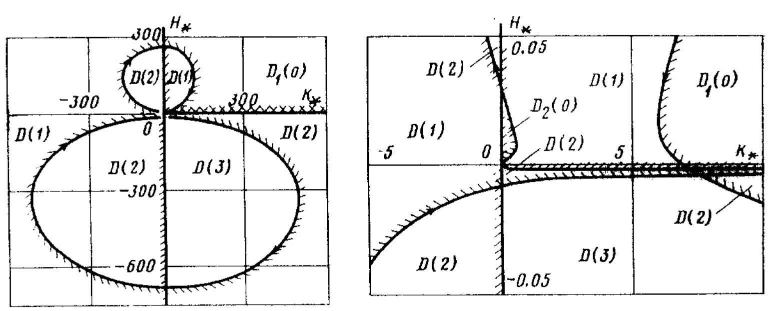

Fig.3

$$K_* - \frac{M}{m}\tau^2 = F_{*\xi}\tau^2, \quad H_*\tau = F_{*\eta}\tau^2; \quad K_* = \frac{K}{m\Omega^2}, \quad H_* = \frac{H}{m\Omega} \tag{5.1}$$

where K_*, H_* are the respective damping the rigidity dimensionless coefficients of the cylinder axis supports.

The dimensionless components $F_{*\xi}, F_{*\eta}$ of force depend only on parameters (3.10). When these parameters are fixed, formulas (5.1) define in the H_*, K_* plane a curve whose points correspond to parameters for which a circular precession of the cylinder is possible. In conformity with the remarks made above that curve separates the plane of parameters of the cylinder axis supports $H_*. K_*$ in region of different degrees of instability. In conformity with /6/ we shall call it the D-curve.

The partitioning of plane H_*, K_* by the D-curve is shown in Fig.3 in the case of $\delta = 0.9, \nu/(\Omega a^2) = 10^{-6}, M/m = 1.68$. The arrows on the D-curve indicate the direction in which parameter τ increases. The D-curve appearing in Fig.3 consists of a regular branch along which parameter τ varies in the interval $(-\infty, +\infty)$. To each value of parameter τ corresponds a point of that curve, and of the singular straight line $K_* = 0$ that corresponds to $\tau = 0$. Presence of the singular straight line is due to that at $\tau = 0$ components of the hydrodynamic force vanish.

The D-curve is conventionally shaded on one side so that the passage from the shaded side to the unshaded corresponds to an increase of the instability degree. In the considered here problem the eigenvalues generally intersect the plane imaginary axis λ one-by-one, not in complex-conjugate pairs. This is due to that when Ω is constant the equations of this problem are not invariant to the change of sign of precession ω; existence of the eigenvalue $\lambda = i\omega$ does not necessarily imply that $\lambda = -i\omega$ is also an eigenvalue. Consequently the passage from the shaded to the unshaded side of the D-curve must generally result in an increase of the degree of instability by unity.

The shading can pass from one side of the D-curve to the other at points where the uniqueness of mapping of the imaginary axis of the plane λ into points of the D-curve is violated /6/. In the considered here problem shading of the D-curve changes only at the regular branch point that corresponds to $\tau = 0$, since only at that value of τ the uniqueness of mapping is violated (the singular straight line in the plane of support of the cylinder axis corresponds to point $\tau = 0$ on the imaginary axis λ of the plane).

The stability region must always contain a point that corresponds to fairly large positive damping coefficients H_*. The zero degree instability region $D_1(0)$ appearing in Fig.3, where regions $D(n)$ of instability of degree n are also shown, conforms to this statement. It is interesting that besides $D_1(0)$ there exists in the neighborhood of zero of H_*, K_* one more stability region $D_2(0)$ (subdivision of that region appears in the right-hand part of Fig.3). Note also that when $H_* = 0$, $\tau = 1$ corresponds to the point on the stability boundary, and this agrees with the known results obtained in /5/ by the phenomenological introduction of internal friction in a rotating rotor.

6. Appendix. Lamb's potentials θ, ψ are not uniquely determined. Indeed, carrying out the substitution $\theta \to \theta + \delta\theta, \psi \to \psi + \delta\psi$, where

$$\frac{\partial}{\partial r}\delta\theta + \frac{1}{r}\frac{\partial}{\partial\varphi}\delta\psi = 0, \quad \frac{1}{r}\frac{\partial}{\partial\varphi}\delta\theta - \frac{\partial}{\partial r}\delta\psi = 0 \tag{6.1}$$

we obtain the same u', v'. Let F be defined in conformity with (3.6). After the change of potentials in conformity with (6.1), we obtain

$$F \to F - 2\Omega\delta\psi - \omega_0 r\frac{\partial}{\partial r}\delta\psi$$

If we now set

$$\delta\psi = r^{-\gamma}\int r^{\gamma-1}g\,dr, \quad \Omega \neq \omega; \quad \delta\psi = \frac{F}{2\Omega}, \quad \Omega = \omega, \quad (g = F/\omega_0, \gamma = 2\Omega/\omega_0) \tag{6.2}$$

the changed potentials θ, ψ are such that F vanishes.

Let us show that Eqs.(6.1) admit $\delta\psi$ of the form (6.2). System (6.1) is solvable for $\delta\theta$, if

$$r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\delta\psi + \frac{\partial^2}{\partial\varphi^2}\delta\psi = 0 \tag{6.3}$$

Substituting (6.2) into (6.3) we obtain $(\Omega \neq \omega)$

$$r\frac{\partial}{\partial r}g - \gamma g + \gamma^2\delta\psi + \frac{\partial^2}{\partial\varphi^2}\delta\psi = 0 \tag{6.4}$$

Integration by parts shows that

$$r^{-\gamma}\int r^{\gamma}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}g\,dr=r\frac{\partial}{\partial r}g-\gamma g+\gamma^2\delta\psi$$

and, consequently, Eq.(6.4) can be represented in the form

$$r^{-\gamma}\int r^{\gamma-1}\left(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}g+\frac{\partial^2}{\partial\varphi^2}g\right)dr=0$$

which is valid, since F satisfies (6.3) for any selected potential. For the same reason $\delta\psi$ defined in (6.2) satisfies (6.3) also when $\Omega=\omega$.

Thus, system (6.1) with $\delta\psi$ defined in (6.2) is solvable, and determines $\delta\theta$ with an accuracy within the additive constant. Because of this, Lamb's potentials can be selected so that $F=0$, with the θ potential still accurate apart the additive constant. Reverting to (3.6), we obtain $G=\text{const}$. The additive constant in potential θ that remains undetermined can be selected so that the constant in the right-hand side of the last equality reduces to zero.

The existence of Lamb's potentials that enable us to reduce system (3.6) to (3.7) has been proved.

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